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ON THE RESPONSE OF AN ELASTIC MEMBRANE TO A TRAVELLING RING LOAD†

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Abstract—Exact solutions in closed-form are presented for the dynamic response of an infinite elastic membrane to a suddenly applied, radially expanding ring load.

INTRODUCTION

In a 1967 paper [1], Kanninen and Florence studied the dynamic response of an infinite elastic membrane to a suddenly applied, radially expanding ring load. To solve the initial boundary value problem they applied the Hankel transform to the radial variable and derived integral representations for the membrane response. However, as the integrals could not be evaluated analytically, they had to resort to numerical integration in order to obtain results on the membrane displacement and velocity. The purpose of this paper is to present exact solutions, in closed-form, for the membrane problem mentioned above.

The membrane problem is solved here by an integral transform method different from that in [1]. We apply the Laplace transform with respect to time t and solve the resulting boundary value problem in the space variable r using the method of Green's function. By the convolution theorem the membrane displacement is expressed in terms of double integrals where the integrands involve the complete elliptic integral of the first kind. The limits of the double integrals depend on whether the load speed v is supersonic or subsonic, i.e. greater or less than the membrane wave speed c, and on r relative to vt and ct. The regions of integration are identified in the various cases. After suitable manipulations and making use of properties of the complete elliptic integral, the limits of integration are changed so that the repeated integrations may be carried out explicitly, with only radicals of quadratic functions being encountered in the integrand at each step.

The solutions for the membrane response will be derived and presented in the next section. Details of the double integrations for the most cases, however, are omitted for brevity. As dynamic loads of the type considered here arise, for example, when sheet explosives are detonated over the surface of a structure, the solutions here may serve to illustrate the response of more general structures to such impulsive loads.

Discussions of the results are given at the end of this paper.

FORMULATION OF THE PROBLEM AND SOLUTIONS

The governing equation for a linear elastic membrane in axisymmetric motion is well known:

$$\rho\frac{\partial^2 w}{\partial t^2}-T\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)=P\delta(r-vt) \tag{1}$$

where $w=w(r,t)$ is the transverse displacement, ρ is the areal density of the membrane, and T is the membrane tension. The right hand side of (1) represents the ring load on the membrane, with P being the load intensity along the load front $r=vt$. We also have the following initial and boundary conditions:

$$w(r,0)=\frac{\partial w}{\partial t}(r,0)=0 \tag{2}$$

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$$\frac{\partial w}{\partial r}(0, t) = w(\infty, t) = 0. \tag{3}$$

We take the Laplace transform of (1) and make use of (2) to obtain

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\hat{w}}{\mathrm{d}r} - \frac{p^2}{c^2}\hat{w} = -N\,\mathrm{e}^{-pr/v} \tag{4}$$

where $\hat{w} = \hat{w}(r, p)$ denotes the Laplace transform of $w(r, t)$, p being the transform parameter, $c = (T/\rho)^{1/2}$ is the membrane wave speed, and $N = P/\rho c^2 v$. The boundary conditions for $\hat{w}(r)$ follow from (3) and are

$$\frac{\mathrm{d}\hat{w}}{\mathrm{d}r}(0) = \hat{w}(\infty) = 0 \tag{5}$$

where and henceforth the dependence of the Laplace transform on p is suppressed.

We express the solutions for $\hat{w}(r)$ as

$$\hat{w}(r) = N\int_0^\infty G(r, r')\,\mathrm{e}^{-pr'/v}\,\mathrm{d}r' \tag{6}$$

where $G(r, r')$ is the "Green's function" [2] for the boundary value problem for $\hat{w}$ and is given by

$$G(r, r') = \begin{cases} r'K_0(pr'/c)I_0(pr/c) & r < r' \\ r'I_0(pr'/c)K_0(pr/v) & r > r' \end{cases} \tag{7}$$

where I_0 and K_0 denote, respectively, the modified Bessel functions, of order zero, of the first and second kind. Substituting (7) into (6) leads to

$$\hat{w}(r) = \hat{w}_1(r) + \hat{w}_2(r) \tag{8}$$

with

$$\hat{w}_1(r) = NK_0(pr/c)\int_0^r r'I_0(pr'/c)\,\mathrm{e}^{-pr'/v}\,\mathrm{d}r' \tag{9}$$

$$\hat{w}_2(r) = NI_0(pr/c)\int_r^\infty r'K_0(pr'/c)\,\mathrm{e}^{-pr'/v}\,\mathrm{d}r'. \tag{10}$$

Taking the Laplace inverse of (8) then yields

$$w(r, t) = w_1(r, t) + w_2(r, t) \tag{11}$$

where

$$w_i(r, t) = \mathscr{L}^{-1}\hat{w}_i(r), \quad i = 1, 2 \tag{12}$$

with $\mathscr{L}^{-1}$ being the inverse Laplace transform operator.

We observe that $\hat{w}_i(r)$ given in (9) and (10) involve products of functions of p. Hence $w_i(r, t)$ may be expressed as convolution integrals of the inverses of such functions. Using the known relations [3]

$$\mathscr{L}^{-1}[\mathrm{e}^{-\alpha p}I_0(\alpha p)] = \begin{cases} \dfrac{1}{\pi\sqrt{(t(2\alpha - t))}} & t < 2\alpha \\ 0 & t > 2\alpha \end{cases} \tag{13}$$

$$\mathscr{L}^{-1}[\mathrm{e}^{\alpha p}K_0(\alpha p)] = \frac{1}{\sqrt{(t^2 + 2\alpha t)}} \tag{14}$$

and with the aid of the translation theorem for Laplace transforms [4], we obtain

$$w_1(r,t)=\frac{N}{\pi}\int\int_{R_1}\frac{r'\,dr'\,dt'}{\sqrt{[(t-t')(2r/c+t-t')]}\sqrt{([t'-(r-r')/c-r'/v][(r+r')/c+r'/v-t'])}} \tag{15}$$

$$w_2(r,t)=\frac{N}{\pi}\int\int_{R_2}\frac{r'\,dr'\,dt'}{\sqrt{[(t-t')(2r/c-t+t')]}\sqrt{([t'-(r'-r)/c-r'/v][t'+(r'+r)/c-r'/v])}} \tag{16}$$

where the regions of integration R_1 and R_2 are defined by

$$R_1\colon 0<t'<t,\quad 0<r'<r,\quad (r-r')/c+r'/v<t'<(r+r')/c+r'/v \tag{17}$$

$$R_2\colon 0<t'<t,\quad r'>r,\quad t-t'<2r/c,\quad t'>(r'-r)/c+r'/v. \tag{18}$$

These regions obviously depend on v relative to c and on r relative to vt and ct. We have the following distinct cases:

(1) $v<c$: R_1 is nonexistent for $r>ct$. For r satisfying $vt<r<ct$, $t(2/c+1/v)^{-1}<r<vt$ and $r<t(2/c+1/v)^{-1}$ respectively, R_1 is as shown in Figs. 1(a)–(c).

(2) $v>c$: R_1 is nonexistent for $r>vt$. For r satisfying $ct<r<vt$, $t(2/c+1/v)^{-1}<r<ct$ and $r<t(2/c+1/v)^{-1}$ respectively, R_1 is as shown in Figs. 2(a)–(c).

(3) R_2 is nonexistent for $r>vt$, and is as shown in Figs. 3(a), (b) respectively for $t(2/c+1/v)^{-1}<r<vt$ and $r<t(2/c+1/v)^{-1}$ regardless of the values of v and c.

In order to find $w_1(r,t)$ and $w_2(r,t)$ from (15) and (16), we first observe that for $v<c$ and $r>ct$, both R_1 and R_2 are nonexistent. The same is true for $v>c$ and $r>vt$. So in these cases $w(r,t)\equiv 0$.

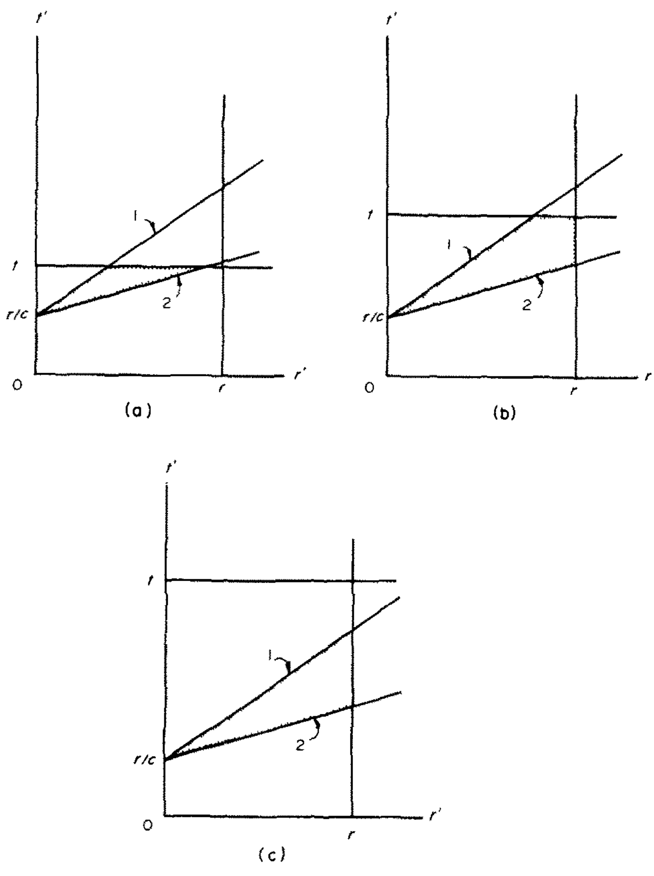

Fig. 1. Regions of integrations for R_1: $v<c$. (a) $vt<r<ct$, (b) $t(2/c+1/v)^{-1}<r<vt$, (c) $r<t(2/c+1/v)^{-1}$. Line 1: $t'=r/c+(1/c+1/v)r'$; Line 2: $t'=r/c+(1/v-1/c)r'$.

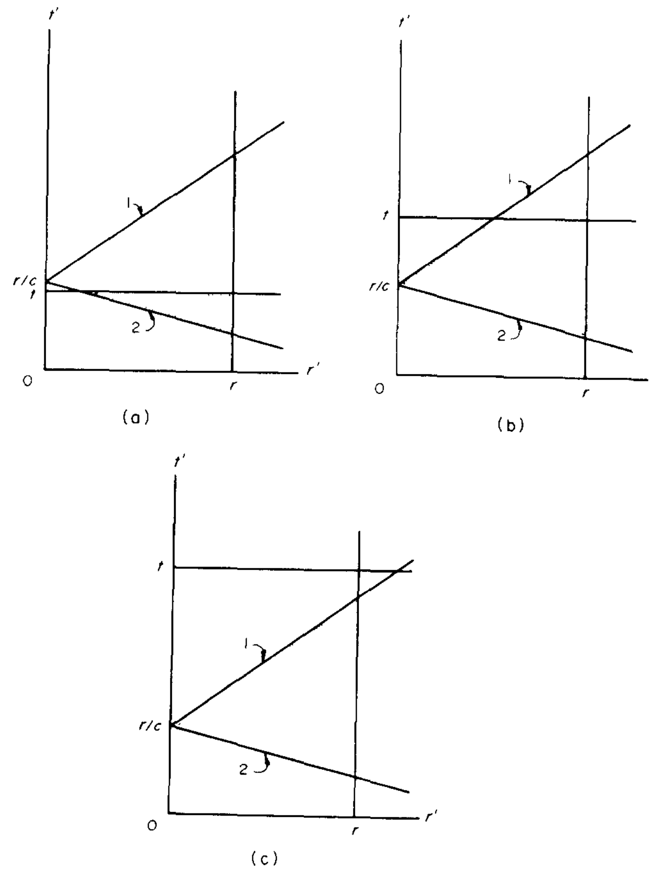

Fig. 2. Regions integration for R_1: $v > c$. (a) $ct < r < ct$, (b) $t(2/c + 1/v)^{-1} < r < ct$, (c) $r < t(2/c + 1/v)^{-1}$. Line 1: $t' = r/c + (1/c + 1/v)r'$; Line 2: $t' = r/c - (1/c - 1/v)r'$.

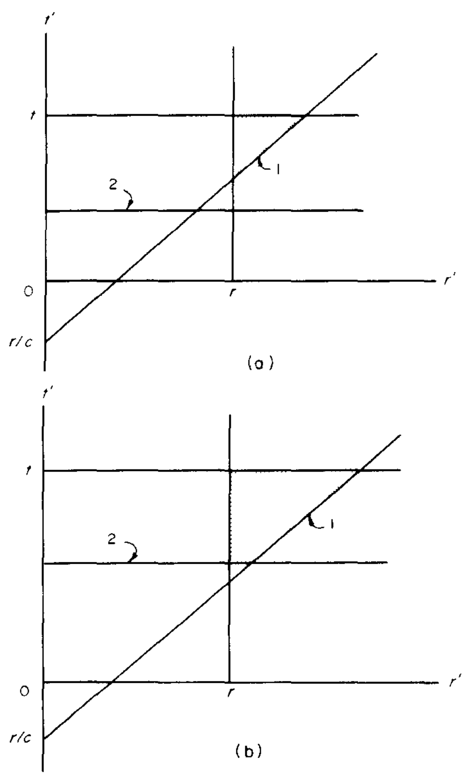

Fig. 3. Regions of integrations for R_2: (a) $t(2/c + 1/v)^{-1} < r < vt$, (b) $r < t(2/c + 1/v)^{-1}$. Line 1: $t' = -r/c + (1/c + 1/v)r'$; Line 2: $t' = t - 2r/c$.

Next consider the case $v<c$ and $vt<r<ct$. Here R_1 is given in Fig. 1(a) while R_2 is nonexistent. Thus $w_2(r,t)\equiv 0$ and from (15) we have

$$w(r,t)=w_1(r,t)=\frac{N}{\pi}\int_{r/c}^{t}\frac{A(r,r',t)\,\mathrm{d}t'}{\sqrt{[(t-t')(2r/c+t-t')]}} \tag{19}$$

where

$$A(r,t,t')=\int_{(t'-r/c)/(1/v+1/c)}^{(t'-r/c)/(1/v-1/c)}\frac{r'\,\mathrm{d}r'}{\sqrt{([t'-(r-r')/c-r'/v][(r+r')/c+r'/v-t'])}}. \tag{20}$$

The expression under the radical sign above is a quadratic in r'. The integration in (20) is elementary and we find

$$A(r,t,t')=\pi\frac{v^2(t'-r/c)}{[1-(v/c)^2]^{3/2}}. \tag{21}$$

Substitution of (21) into (19) yields

$$w(r,t)=\frac{Nv^2}{[1-(v/c)^2]^{3/2}}\int_{r/c}^{t}\frac{(t'-r/c)\,\mathrm{d}t'}{\sqrt{[(t-t')(2r/c+t-t')]}}. \tag{22}$$

We note that the expression under the radical sign above is a quadratic in t'. Carrying out the t'-integration we finally obtain in this case

$$w(r,t)=\frac{Nv^2t}{[1-(v/c)^2]^{3/2}}\{\cosh^{-1}(ct/r)-[1-(r/ct)^2]^{1/2}\}. \tag{23}$$

We now consider the case $v>c$ and $ct<r<vt$, for which R_1 is given in Fig. 2(a) and R_2 in Fig. 3(a). We express $w_1(r,t)$ and $w_2(r,t)$ in (15) and (16) as

$$w_1(r,t)=\frac{N}{\pi}\int_{(r/c-t)/(1/c-1/v)}^{r} r'B_1(r,r',t)\,\mathrm{d}r' \tag{24}$$

$$w_2(r,t)=\frac{N}{\pi}\int_{r}^{(r/c+t)/(1/c-1/v)} r'B_2(r,r',t)\,\mathrm{d}r' \tag{25}$$

where

$$B_1(r,r',t)=\int_{\delta_1}^{\gamma_1}\frac{\mathrm{d}t'}{\sqrt{[(\alpha_1-t')(\beta_1-t')(\gamma_1-t')(t'-\delta_1)]}} \tag{26}$$

$$B_2(r,r',t)=\int_{\beta_2}^{\alpha_2}\frac{\mathrm{d}t'}{\sqrt{[(\alpha_2-t')(t'-\beta_2)(t'-\gamma_2)(t'-\delta_2)]}} \tag{27}$$

with

$$\alpha_1=2r/c+t,\quad \beta_1=(r+r')/c+r'/v,\quad \gamma_1=t,$$
$$\delta_1=(r-r')/c+r'/v \tag{28}$$

$$\alpha_2=t,\quad \beta_2=-(r-r')/c+r'/v,\quad \gamma_2=t-2r/c,$$
$$\delta_2=-(r+r')/c+r'/v. \tag{29}$$

We also introduce the integral $B_3(r,r',t)$ defined by

$$B_3(r,r',t)=\int_{\delta_3}^{\gamma_3}\frac{\mathrm{d}t'}{\sqrt{[(\alpha_3-t')(\beta_3-t')(\gamma_3-t')(t'-\delta_3)]}} \tag{30}$$

with

$$\alpha_3 = t + (r + r')/c - r'/v, \quad \beta_3 = 2r/c, \quad \gamma_3 = t - (r - r')/c - r'/v, \quad \delta_3 = 0. \tag{31}$$

We observe that $\alpha_i > \beta_i > \gamma_i > \delta_i$ for $i = 1, 2, 3$ over the respective regions of integration.

The integrals $B_i(r, r', t)$, $i = 1, 2, 3$, are all identical since

$$\begin{aligned} B_i(r, r', t) &= \frac{2}{\sqrt{[(\alpha_i - \gamma_i)(\beta_i - \delta_i)]}} K\left(\frac{(\alpha_i - \beta_i)(\gamma_i - \delta_i)}{(\alpha_i - \gamma_i)(\beta_i - \delta_i)}\right) \\ &= \frac{c}{\sqrt{(rr')}} K\left(\frac{c^2[t + (r - r')/c - r'/v][t - (r - r')/c - r'/v]}{4rr'}\right) \quad i = 1, 2, 3 \end{aligned} \tag{32}$$

where K denotes the complete elliptic integral of the first kind[5].

We now combine (24) and (25) to form a single r'-integral for $w(r, t)$ and then express it as the difference of two integrals

$$\begin{aligned} w(r, t) &= \frac{N}{\pi} \int_{(r/c - t)/(1/c - 1/v)}^{(r/c + t)/(1/c + 1/v)} r' B_3(r, r', t) \, dr' \\ &= z_1(r, t) - z_2(r, t) \end{aligned} \tag{33}$$

with

$$z_1(r, t) = \frac{N}{\pi} \int_0^{(r/c + t)/(1/c + 1/v)} r' B_3(r, r', t) \, dr' \tag{34}$$

$$z_2(r, t) = \frac{N}{\pi} \int_0^{(r/c - t)(1/c - 1/v)} r' B_3(r, r', t) \, dr'. \tag{35}$$

The double integration for $z_1(r, t)$ is over a triangular region in the (r', t') plane with vertices at $(0, 0)$, $([r/c + t]/[1/c + 1/v], 0)$ and $(0, r/c + t)$. Upon interchanging the order of integration we have

$$z_1(r, t) = \frac{N}{\pi} \int_0^{r/c + t} \frac{A_1(r, t, t') \, dt'}{\sqrt{[t'(2r/c - t')]}} \tag{36}$$

where

$$A_1(r, t, t') = \int_0^{(r/c + t - t')/(1/c + 1/v)} \frac{r' \, dr'}{\sqrt{([r/c + t - t' + r'(1/c - 1/v)][r/c + t - t' - r'(1/c + 1/v)])}}. \tag{37}$$

The double integration for $z_2(r, t)$ as given in (35) is over a trapezoid in the (r', t') plane. We note that for $0 < r' < [(r/c - t)/(1/c - 1/v)]$, the values α_3, β_3, γ_3, δ_3 as defined in (31) satisfy $\beta_3 > \alpha_3 > \gamma_3 > \delta_3$. We replace $B_3(r, r', t)$ in (35) by the identical elliptic integral $B_4(r, r', t)$ defined by

$$B_4(r, r', t) = \int_{\beta_4}^{\alpha_4} \frac{dt'}{\sqrt{[(\alpha_4 - t')(t' - \beta_4)(t' - \gamma_4)(t' - \delta_4)]}} \tag{38}$$

with $\alpha_4 = \beta_3$, $\beta_4 = \alpha_3$, $\gamma_4 = \gamma_3$, $\delta_4 = \delta_3$. It is seen that $B_4(r, r', t)$ reduces to (32) for $i = 4$. The double integration for $z_2(r, t)$ is now over a triangular region with vertices at $(0, r/c + t)$, $(0, 2r/c)$ and $([r/c - t]/[1/c - 1/v], 2r/c)$. Upon interchanging the order of integration we have

$$z_2(r, t) = \frac{N}{\pi} \int_{r/c + t}^{2r/c} \frac{A_2(r, t, t') \, dt'}{\sqrt{[(2r/c - t')t']}} \tag{39}$$

where

$$A_2(r,t,t') = \int_0^{(t'-t-r/c)/(1/c-1/v)} \frac{r'\,\mathrm{d}r'}{\sqrt{([t'-t-r/c-r'(1/c-1/v)][t'-t-r/c+r'(1/c+1/v)])}}. \quad (40)$$

All the integrations above can now be performed explicitly, with the integrands involving only radicals of quadratics of the variable of integration. From (37) and (40) we obtain

$$A_1(r,t,t') = c_1(r/c+t-t') \quad (41)$$

$$A_2(r,t,t') = c_2(-r/c-t+t') \quad (42)$$

where c_1 and c_2 are constants given by

$$c_1 = \frac{v^2}{[(v/c)^2-1]^{3/2}}\{[(v/c)^2-1)^{1/2} - \pi/2 + \sin^{-1}(c/v)\} \quad (43)$$

$$c_2 = \frac{v^2}{[(v/c)^2-1]^{3/2}}\{[(v/c)^2-1]^{1/2} + \pi/2 + \sin^{-1}(c/v)\}. \quad (44)$$

Substituting (41) into (36) and (42) into (39) and performing the t'-integrations we then have

$$z_1(r,t) = \frac{N}{\pi}c_1 t\{[(r/ct)^2-1]^{1/2} + \sin^{-1}(ct/r) + \pi/2\} \quad (45)$$

$$z_2(r,t) = \frac{N}{\pi}c_2 t\{[(r/ct)^2-1]^{1/2} + \sin^{-1}(ct/r) - \pi/2\}. \quad (46)$$

From (33) we finally obtain

$$w(r,t) = \frac{Nv^2t}{[(v/c)^2-1]^{3/2}}\{[(v/c)^2-1]^{1/2} + \sin^{-1}(c/v) - [(r/ct)^2-1]^{1/2} - \sin^{-1}(ct/r)\}. \quad (47)$$

Details for the remaining cases are similar to those given here and are omitted. We now summarize the solutions for $w(r,t)$ normalized with respect to $v_0 t$, where $v_0 = P/\rho v$. Also, in what follows $\beta = v/c$, $\alpha = r/ct$.

I. $v < c$ $(\beta < 1)$

$$\frac{w(r,t)}{v_0 t} = \begin{cases} 0 \quad r > ct \quad (\alpha > 1) \\ \dfrac{\beta^2}{(1-\beta^2)^{3/2}}[\cosh^{-1}(1/\alpha) - (1-\alpha^2)^{1/2}] \\ vt < r < ct \quad (\beta < \alpha < 1) \\ \dfrac{\beta^2}{(1-\beta^2)^{3/2}}[\cosh^{-1}(1/\beta) - (1-\beta^2)^{1/2}] \\ \quad r < vt \quad (\alpha < \beta). \end{cases} \quad (48)$$

II. $v > c$ $(\beta > 1)$

$$\frac{w(r,t)}{v_0 t} = \begin{cases} 0 \quad r > vt \quad (\alpha > \beta) \\ \dfrac{\beta^2}{(\beta^2-1)^{3/2}}[(\beta^2-1)^{1/2} + \sin^{-1}(1/\beta) \\ -(\alpha^2-1)^{1/2} - \sin^{-1}(1/\alpha)] \end{cases} \quad (49)$$

$$
\begin{array}{l}
ct < r < vt \quad (1 < \alpha < \beta) \\
\dfrac{\beta^2}{(\beta^2-1)^{3/2}}[(\beta^2-1)^{1/2} + \sin^{-1}(1/\beta) - \pi/2] \\
\quad r < ct \quad (\alpha < 1).
\end{array}
$$

Also, when $v \to c$, the solutions in (48) and (49) both tend to the following limit

III. $v = c$ $(\beta = 1)$

$$
\frac{w(r,t)}{v_0 t} = \begin{cases} 0 & r > ct \quad (\alpha > 1) \\ \dfrac{1}{3} & r < ct \quad (\alpha < 1). \end{cases} \tag{50}
$$

We also present the solutions for the membrane velocity normalized with respect to v_0, which are obtained from differentiating $w(r, t)$ with respect to time.
I. $v < c$ $(\beta < 1)$

$$
\frac{1}{v_0}\frac{\partial w}{\partial t}(r,t) = \begin{cases} 0 \quad r > ct \quad (\alpha > 1) \\ \dfrac{\beta^2}{(1-\beta^2)^{3/2}} \cos h^{-1}(1/\alpha) \\ \quad v < r < ct \quad (\beta < \alpha < 1) \\ \dfrac{\beta^2}{(1-\beta^2)^{3/2}} [\cos h^{-1}(1/\beta) - (1-\beta^2)^{1/2}] \\ \quad r < vt \quad (\alpha < \beta). \end{cases} \tag{51}
$$

II. $v > c$ $(\beta > 1)$

$$
\frac{1}{v_0}\frac{\partial w}{\partial t}(r,t) = \begin{cases} 0 \quad r > vt \quad (\alpha > \beta) \\ \dfrac{\beta^2}{(\beta^2-1)^{3/2}} [(\beta^2-1)^{1/2} + \sin^{-1}(1/\beta) - \sin^{-1}(1/\alpha)] \\ \quad ct < r < vt \quad (1 < \alpha < \beta) \\ \dfrac{\beta^2}{(\beta^2-1)^{3/2}} [(\beta^2-1)^{1/2} + \sin^{-1}(1/\beta) - \pi/2] \\ \quad r < ct \quad (\alpha < 1). \end{cases} \tag{52}
$$

III. $v = c$ $(\beta = 1)$

$$
\frac{1}{v_0}\frac{\partial w}{\partial t}(r,t) = \begin{cases} 0 & r > ct \quad (\alpha > 1) \\ \dfrac{1}{3} & r < ct \quad (\alpha < 1). \end{cases} \tag{53}
$$

DISCUSSIONS

We can now make some observations on the membrane response. The ring load on the membrane starts at the origin at $t = 0$ and moves outwardly at the speed v. Since disturbances in the membrane propagate with the speed c, we have, at any $t > 0$, the "load front" $r = vt$ and the "wave front" $r = ct$. Obviously this latter front precedes the former in the subsonic case $v < c$ and vice versa in the supersonic case $v > c$.

The solutions given above reveal that the membrane response takes different forms in the regions separated by the load and wave fronts and depends on whether v is less or greater than c.

In agreement with the findings in [1] we see that the membrane response is identically zero outside the outer front in either the subsonic or the supersonic case. Also, in either case, the membrane has a uniform velocity, independent of r and t, inside the inner front. This last observation was suggested by the numerical result in [1] but was not established analytically there. It is interesting to note that Kanninen and Florence[1] were able to derive closed-form solutions for the membrane velocity at the center of the membrane and their results agree with the constant velocities given here.

We also see that, except for the case $v = c$, the membrane displacement is continuous across both fronts and the membrane velocity, while being continuous across the wave front, is discontinuous across the load front. The explicit solutions given for the region bounded by $r = vt$ and $r = ct$ show that the membrane displacement increases monotonically from the outer front to the inner front. Also, the membrane velocity increases from the outer front to the inner front in the subsonic case while decreases from the outer front to the inner front in the supersonic case. The maximum membrane velocity at any t always occurs near the load front—immediately preceding the load front in the subsonic case and immediately trailing the load front in the supersonic case. We also remark that the discontinuity in the membrane velocity across the load front is expected from the dynamic equation for the membrane, which is of the second order in t and has the load term in the form of a delta function along the load front $r = vt$. An integration with respect to t across $r = vt$ will result in a finite jump in the membrane velocity across the load front. We should expect this to hold true for more general elastic structures.

The solution for the membrane displacement w is not continuous across $r = vt = ct$ when $v = c$, and is thus not physically meaningful. As v tends to c sharp gradients of w in r are developed in the region bounded by $r = vt$ and $r = ct$. Since large $\partial w/\partial r$ corresponds to large membrane strain we should not expect the linear membrane theory to hold in the limiting case as v tends to c.

The characteristic velocity v_0 which we used in the normalization of the membrane displacement and velocity has the physical meaning that it is the velocity which would be attained by the membrane elements if they were all disconnected[1]. Kanninen and Florence[1] were primarily interested in the simulation problem, i.e. to obtain criteria for justifying the approximation of the moving load by a uniform simultaneous impulse over the region inside the load front and concluded that good simulation is achieved in the supersonic case for large values of β. This is also obvious from (52), which shows as β becomes large, the ratio $\partial w/\partial t$ to v_0 tends to unity in $r < vt$.

Numerical results are now easily obtained and are presented in Figs. 4–6 for $\beta = 0.707$, 1.0 and 4.0. These results are similar to those given in [1].

We remark finally that the solutions obtained here may find useful applications in other problems involving similar moving sources as many engineering and physical phenomena are

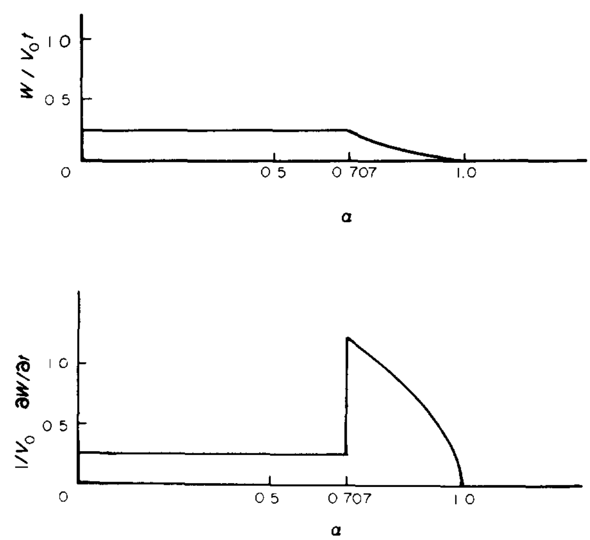

Fig. 4. Membrane displacement and velocity. $\beta = 0.707$.

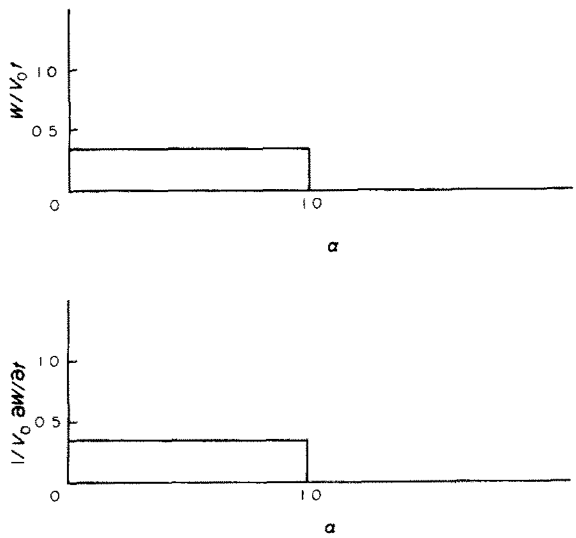

Fig. 5. Membrane displacement and velocity. $\beta = 1.0$.

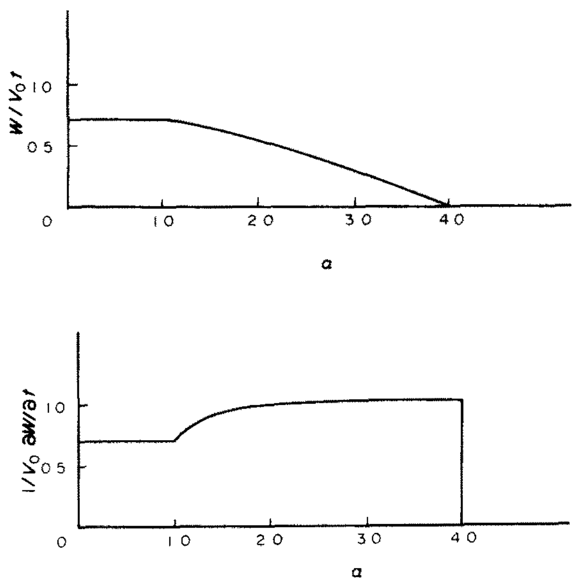

Fig. 6. Membrane displacement and velocity. $\beta = 4.0$.

governed by the same wave equation. Also the explicit solutions here are useful in nonlinear membrane problems as they provide the leading terms in perturbation expansions. They are also useful in checking the accuracy of possible numerical schemes developed for treating more general moving load problems.

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